Normal subgroup

Let G be a multiplicative abelian group and H be a subgroup of G. If x be any element of G, then Hx is the right coset of H in G and xH is a left coset of H in G. Now, G is abelian $Hx = xH \; \forall x \in G$. But it may so happen that G is not abelian yet it may have a sub-group G such that for any $x \in G$, $Hx = xH$. This gives rise to a class of sub-groups of G which comes under the class of normal-sub groups.

A sub-group H of a group G is said to be a normal sub-group of G, if for every $x \in G$ and for every $h \in H$, $xhx^{-1} \in H$.

This definition is equivalent to saying that H is a normal sub-group of the group G, if and only if $xHx^{-1} \subset H \; \forall x \in G$.

If $xHx^{-1} = H \; \forall x \in G$, then truly $xHx^{-1} \subset H$, and so by definition, H is a normal sub-group of G.

In this case we have $(xHx^{-1})x = Hx$
or $xH = Hx \; \forall x \in G$.
Hence each left coset xH is the right coset Hx.

Again let G be an abelian group and H be a subgroup of G. G being abelian, we have $xH = Hx \; \forall x \in G$. Thus H is a normal sub-group of G.

Further, since every cyclic group is abelian,

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We observe that every sub-group of a cyclic group is normal.

Note: 1) A normal sub-group is also called invariant or self conjugate sub-group.

2) We have seen earlier that $3\mathbb{Z}$ is a sub-group of the group $(\mathbb{Z}, +)$. Since $(\mathbb{Z}, +)$ is abelian, $3\mathbb{Z}$ is a normal sub-group.

Theorem 1 H is a normal sub-group of G if and only if $xHx^{-1} = H$, $\forall h \in H$ and $\forall x \in G$

Proof: Let H is a normal sub-group of G.

Then $Hx = xH \quad \forall x \in G$

$\Rightarrow x^{-1}Hx = x^{-1}(xH) = (x^{-1}x)H = eH = H$

Conversely, let $x^{-1}Hx = H \quad \forall x \in G$.

Then $x(x^{-1}Hx) = xH$

$\Rightarrow (xx^{-1})Hx = xH$

$\Rightarrow eHx = xH$

$\Rightarrow Hx = xH$

Hence H is normal.

Theorem 2 A sub-group H of a group G is normal in G if and only if $x^{-1}hx \in H$ for all $h \in H$, $x \in G$.

(Home work)